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G_2 quivers

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Abstract: We present, in explicit matrix representation and a modernity befitting the community, the classification of the finite discrete subgroups of G_2 and compute the McKay quivers arising therefrom. Of physical interest are the classes of $\mathcal{N} = 1$ gauge theories descending from M-theory and of mathematical interest are possible steps toward a systematic study of crepant resolutions to smooth G_2 manifolds as well as generalised McKay Correspondences. This writing is a companion monograph to hep-th/9811183 and hep-th/9905212, wherein the analogues for Calabi-Yau three- and four-folds were considered.

Keywords: M-Theory, D-branes, Brane Dynamics in Gauge Theories, Discrete and Finite Symmetries.

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Contents

1. Introduction

As Ricci-flat threefolds of SU(3) holonomy were upon the centre stage in the compactification of string theory in the past two decades, so too have phenomenological considerations lead us in the past two years to focus on M-theory compactification to real sevenfolds of G_2 holonomy [1]–[6]. And thus compounded upon our tremendous attention on Calabi-Yau and Kähler manifolds, both physically and mathematically, is our revival of interest on special holonomy [7, 8] in super string theory and indeed, on real algebraic geometry. Inspired thereby, an impressive collection of excellent works appeared in quick succession within the community.

Yet, our inherent difficulty with the explicit construction of G_2 manifolds has been one fundamental limitation and so far very few examples of compact G_2 sevenfolds are known [9, 10].

A classic theme in the Calabi-Yau case has been that whereas understanding string theory on arbitrary members in the space of these manifolds is substantially challenging, one could gain great insight by moving to particular points in the moduli space and notably, to the singular Calabi-Yau's. In other words, we model our compact (projective) variety with local, non-compact (affine) algebraic singularities (cf. [11] for some review in this context). For example, the Calabi-Yau twofold K3 can be regarded locally as resolutions of the ALE orbifold $\mathbb{C}^2/\Gamma$ for Γ some discrete finite subgroup — of ADE type — of SU(2), the holonomy of K3.

The advent of the technology of D-brane probes on such singularities [27] has brought about systematic methods [28] for constructing low-energy effective gauge theories of various supersymmetry, matter content and interactions. The case of K3 was considered in [29], that of local orbifolds of Calabi-Yau threefolds, in [21, 23] and that of fourfolds in [24].

A natural question then arises as to whether we could do the same, as we did for holonomy $\mathrm{SU}(n = 2, 3, 4)$, for local G_2 and study the discrete finite subgroups Γ of G_2 and hence orbifolds of the form $\mathbb{R}^7/\Gamma$. Yet, *caveat emptur*; for though with the extensive machinery of complex algebraic geometry, canonical Gorenstein singularities of the form $\mathbb{C}^2/(\Gamma \subset \mathrm{SU}(2))$, and $\mathbb{C}^3/(\Gamma \subset \mathrm{SU}(3))$ for abelian Γ have been intensively studied and found to admit crepant resolutions, the same certainly cannot be said for orbifolds of $\mathbb{R}^7$. In other words, though in the Calabi-Yau case there are well-studied resolutions to smooth Ricci-flat manifolds from the said orbifolds (which in string theory correspond to well-defined states which become massless at the singular limit), our present lack of more tools of real algebraic geometry, let alone real resolutions, hinders a complete understanding of such G_2 orbifolds.

Nevertheless, much nice works have been on non-compact G_2 manifolds (cf. [16] and references therein) and in particular on D-brane probes on G_2 orbifolds [12]–[15]. Examples of abelian orbifolds have been detailed in [12, 15] while elegant extensions of ADE singularities to real dimension seven have been addressed in [13, 14]. Therefore naturally does arise a present want which desires a immediate supplement and indeed for which [12] has kindly beckoned: the classification of the discrete subgroups of G_2.

And thus is the purpose of the present writing. As a companion monograph to [21, 24], we have transcribed some known results, collected from mathematical works seemingly obscure to the physics literature [17]–[19] and recast them into a compilation explicit in representation, feasible to computation and abundant in tabulation. These shall constitute sections 2 and 3. We then calculate and draw what we call "G_2 quivers" from this data in section 4 and discuss implications thereof.

2. Some preparatory remarks and nomenclature

2.1 Reducibility and primitivity

We are concerned with finite discrete subgroups of Lie groups and are thus confined to the study of linear transformations, manifesting as matrix groups acting upon vector spaces. Indeed, standard in the mathematical literature is the following terminology which further categorises such groups. The reader is referred to the excellent monographs [25] and [26], (or to [24] in a context more immediate to this paper), for further details.

Essentially, a linear transformation group Γ is called Intransitive or **Reducible** if it is block-diagonalizable and Transitive or **Irreducible** otherwise. The Irreducible Γ can be further divided into the **Primitive** and **Imprimitive**, where the imprimitive can still have blocks of zeros while the primitive groups generically have no zero entries and are the fundamental building blocks in the classification.

The usual scheme of classification of the subgroups of Lie groups is over the field $\mathbb{C}$ whereas for obvious physical reasons we are interested in 7-manifolds over $\mathbb{R}$ and hence discrete subgroups of $G_2(\mathbb{R})$. Henceforth by G_2 we shall mean $G_2(\mathbb{R})$. Indeed the classification in light of the categorisations [17]–[19] has been performed for $G_2(\mathbb{C})$ and we beg the reader to take heed that in the ensuing reducibility etc. refer to the groups over $\mathbb{C}$. Nevertheless we can still refer to these groups under their present categories, since any subgroup of $G_2(\mathbb{C})$, being compact, is actually contained in a maximal compact subgroup of $G_2(\mathbb{C})$ and is hence conjugate to a subgroup $G_2(\mathbb{R})$ [18] and whence the classifications coincide in any event. Of course, we shall be careful to take appropriate involutions to ensure that our matrix groups are indeed real in that they have generators in $\mathrm{GL}(7,\mathbb{R})$ and hence have a real 7-dimensional irrep as reflected in the character tables.

Unless otherwise stated, we adhere to the following nomenclature throughout the writing. By Lie groups of finite type we mean discrete finite groups which are the corresponding continuous Lie groups defined over some Galois field. Thus for example $\mathrm{GL}(n;q)$ is the general linear group $\mathrm{GL}(n)$ over the field $\mathbb{F}_q$; it is thus the endomorphism for the vector space $\mathbb{F}_q^{\oplus n}$. By $\Gamma := \langle\{a_i\}\rangle$ we shall mean that the finite group Γ is generated by elements (matrices) a_i.

2.2 Automorphisms of the octonions

Let us first recall some rudimentary facts concerning G_2 in light of its linear transformational properties [17]. The Octonions $\mathbb{O}$ is a real non-associative division algebra. In particular, it is a 7-dimensional vector space over $\mathbb{R}$, endowed with basis $e_0 = \mathrm{Id}$ as well as $\{e_{i=1,\ldots,7}\}$ satisfying

$$e_i^2 = -1\,, \qquad e_i e_j = e_k \quad \text{for} \quad (i,j,k) = (1+r, 2+r, 3+r) \bmod 7\,.$$

On this vector space, a natural quadratic form exists for any element $x \in \mathbb{O} = \sum\limits_{i=0}^{7} a_i e_i$ (where $a_i \in \mathbb{R}$), namely $Q(x) = \sum_i a_i^2$. Thenceforth the following bilinear and trilinear forms $b(\_,\_)$ and $t(\_,\_,\_)$ can be established for $x,y,z \in \mathbb{O}$:

$$\begin{aligned} b(x,y) &= \frac{1}{2}\left(Q(x+y) - Q(x) - Q(y)\right) \\ t(x,y,z) &= b(xy,z)\,. \end{aligned} \tag{2.1}$$

In fact, any automorphism of $\mathbb{O}$ preserves these above forms. The group $\mathrm{Aut}(\mathbb{O})$ of these automorphisms is isomorphic to $G_2(\mathbb{R})$, or what we shall refer to[1] as G_2.

It is into the discrete finite subgroups of this automorphism group, as linear transformations of the real vector space $\mathbb{O}$, that this writing shall delve.

3. The classification of the discrete finite subgroups of G_2

The classification, in its original form, has been existent in the mathematical literature for some time [17, 18]. Such a result of Wales-Cohen has been transposed into modernity by Griess in [19]. Our first task then, before moving on to quivers and gauge theories, is to recast yet again, all these marvelous results, from their perhaps abstruse guise, to a more tangible form, whose concrete realisation as matrix groups are explicit.

[1] $G_2(\mathbb{C})$ is thus $\mathrm{Aut}(\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C})$.

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3.1 Reducible

As with all classifications of these discrete finite subgroups of Lie groups (cf. [25, 26]), the reducible groups are always direct and semi-direct products of Lie subgroups of the parent. In the case of G_2, all these reducible are constructable from the finite discrete subgroups of SU(2) and SU(3) [17, 9]. Explicit representations of these infinite series follow along the lines of the ZD-type groups for SU(3) in [22, 23], composed of two non-commuting pieces, viz. the Z and the D of SU(2) $\hookrightarrow$ SU(3), with their generators appropriately concatenated. So likewise could we do so for G_2.

Now in order to preserve the automorphism structure of $\mathbb{O}$ and reality of our 7-dimensional representation, the denouement is that [17] only subgroups of (a) SU(2)×SU(2) and (b) SU(3) are allowed. Therefore, the reducible finite subgroups of G_2 are quite well-known, as has been considered in for example [12, 15] and easily extended from the famous ADE subgroups of SU(2) [20] as well as those of SU(3) [25, 26, 21]. Therefore upon these irreducibles let us not dwell.

3.2 Irreducible imprimitive

The heart of the classification lies in the irreducibles, which in some sense reflect the intricacies of the structure of Aut($\mathbb{O}$). There are in all 7 of these, 2 imprimitive and 5 primitive. To the particulars of these 7 exceptionals let us now turn. The ensuing computations are done with the extensive aid of [30] to whose writers we are forever indebted.

The first irreducible imprimitive we shall call II_1; it has the following generators

$$II_1 := \left\langle \begin{pmatrix} 0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ -1&-1&-1&-1&-1&-1&-1\\ 0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0\\ -1&-1&-1&-1&-1&-1&-1 \end{pmatrix} \right\rangle . \tag{3.1}$$

We see of course that II_1 is indeed 7-dimensional over the reals. This group is in fact none other than the projective special linear group over the finite field $\mathbb{F}_7$, PSL(2; 7), which is isomorphic to another Lie group of finite type, viz. GL(3; 2). Moreover, II_1 is a group of order 168 and is in fact isomorphic to the familiar exceptional group Σ_{168} of SU(3) [26, 21].

The character table for II_1 is computed as follows:

	1	21	24	24	42	56
Γ_1	1	1	1	1	1	1
Γ_2	3	−1	w	$\bar{w}$	1	0
Γ_3	3	−1	$\bar{w}$	w	1	0
Γ_4	6	2	−1	−1	0	0
Γ_5	7	−1	0	0	−1	1
Γ_6	8	0	1	1	0	−1

$$w := \frac{-1-\sqrt{7}}{2}.$$

Moving on to the next in the irreducible imprimitives, we have II_2, which is generated by

$$II_2 := \left\langle \begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0\\ 1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&-1&0&0&0&0&0\\ -1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&-1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&-1 \end{pmatrix} \right\rangle . \quad (3.2)$$

Now II_2 is a group of order 1344 and is in fact a central extension of II_1 in the sense that $II_2/\mathbb{Z}_2^3 \cong II_1$. It character table is as follows:

	1	7	42	42	84	168	168	192	192	224	224
Γ_1	1	1	1	1	1	1	1	1	1	1	1
Γ_2	3	3	−1	−1	−1	1	1	w	$\bar{w}$	0	0
Γ_3	3	3	−1	−1	−1	1	1	$\bar{w}$	w	0	0
Γ_4	6	6	2	2	2	0	0	−1	−1	0	0
Γ_5	7	−1	−1	3	−1	−1	1	0	0	−1	1
Γ_6	7	7	−1	−1	−1	−1	−1	0	0	1	1
Γ_7	7	−1	3	−1	−1	1	−1	0	0	−1	1
Γ_8	8	8	0	0	0	0	0	1	1	−1	−1
Γ_9	14	−2	2	2	−2	0	0	0	0	1	−1
Γ_{10}	21	−3	−3	1	1	1	−1	0	0	0	0
Γ_{11}	21	−3	1	−3	1	−1	1	0	0	0	0

with w as in the characters for II_1.

Thus concludes the irreducible imprimitives.

3.3 Irreducible primitive

We now present the irreducible primitives, which, as was mentioned in the preliminary remarks, are the true fundamental building blocks. There are 5 of these groups, of substantial size and we beseech the reader's patience.

The first primitive we shall call IP_1, with the rather complicated generators

$$IP_1 := \left\langle \frac{1}{c}\begin{pmatrix} 23b1+22b2 & 31b1+41b2 & -15b1-12b2 & 7b1-7b2 & 18b1+27b2 & 9b2 & 14b1+22b2\\ -4b1-5b2 & -23b1-40b2 & 12b1-3b2 & 7b1+20b2 & -9b1-27b2 & -c/2 & -22b1-32b2\\ 0&0&0&0&0&0&c\\ -19b1-17b2 & -8b1-37b2 & -15b1-21b2 & 4b1+23b2 & -45b1-54b2 & 27b1+18b2 & -10b1-26b2\\ -5b1-4b2 & -13b1-23b2 & 15b1+12b2 & -7b1+7b2 & -9b2 & -9b2 & 4b1-4b2\\ 2b1-2b2 & -2b1+2b2 & 12b1+6b2 & -8b1-10b2 & 0 & 0 & 2b1-2b2\\ 5b1+4b2 & 13b1+23b2 & 3b1+6b2 & 7b1-7b2 & 18b1+27b2 & -18b1-9b2 & 14b1+22b2 \end{pmatrix}, \right.$$
$$\left. \frac{1}{c}\begin{pmatrix} 5b1+4b2 & 13b1+23b2 & -15b1-12b2 & 7b1-7b2 & 9b2 & 9b2 & -4b1+4b2\\ 17b1+10b2 & b1-19b2 & 3b1+6b2 & 13b1+23b2 & -9b2 & -9b2 & 8b1-8b2\\ 18b1+9b2 & 27b1+54b2 & 9b2 & 9b1-18b2 & 9b1+27b2 & -c/2 & 18b1+36b2\\ -5b1-4b2 & -31b1-41b2 & 15b1+12b2 & -7b1+7b2 & -9b2 & -9b2 & -14b1-22b2\\ -19b1-8b2 & -17b1-19b2 & 3b1+6b2 & -5b1+5b2 & -36b1-27b2 & 18b1+9b2 & -10b1-8b2\\ -19b1-17b2 & 10b1+35b2 & -15b1-3b2 & -14b1-31b2 & -45b1-18b2 & 27b1+18b2 & -10b1+10b2\\ b1-b2 & 8b1+b2 & -3b1-15b2 & -4b1-5b2 & 9b1 & -9b1 & -8b1-10b2 \end{pmatrix} \right\rangle$$

with

$$\begin{aligned} x &:= \cos\left(\frac{2\pi}{13}\right), \qquad c := \frac{9}{1+2x}\,;\\ b1 &:= 2-9x-2x^2+24x^3-16x^5\,, \qquad \text{and}\\ b2 &:= -3+14x+4x^2-44x^3+32x^5\,. \end{aligned} \quad (3.3)$$

Now IP_1 is nothing but the group PSL(2;13), of order 1092. Its characters are given below:

	1	84	84	91	156	156	156	182	182
Γ_1	1	1	1	1	1	1	1	1	1
Γ_2	7	p	q	-1	0	0	0	-1	1
Γ_3	7	q	p	-1	0	0	0	-1	1
Γ_4	12	-1	-1	0	r_1	r_2	r_3	0	0
Γ_5	12	-1	-1	0	r_2	r_3	r_1	0	0
Γ_6	12	-1	-1	0	r_3	r_1	r_2	0	0
Γ_7	13	0	0	1	-1	-1	-1	1	1
Γ_8	14	1	1	-2	0	0	0	1	-1
Γ_9	14	1	1	2	0	0	0	-1	-1

with $p := (1-\sqrt{13})/2$, $q := (1+\sqrt{13})/2$ and $r_{1,2,3}$ the three roots of the cubic equation $1 - 2r - r^2 + r^3 = 0$.

The next in the family is IP_2, generated by

$$IP_2 := \left\langle \begin{pmatrix} -1/3 & 0 & 1 & -4/3 & 1/3 & -1 & 5/3 \\ -3 & -2 & 2 & -1 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -4/3 & 0 & 1 & 2/3 & -5/3 & 1 & 2/3 \\ 4 & 3 & -3 & 2 & 1 & 2 & -4 \\ -1 & 0 & 1 & -1 & 1 & -1 & 2 \end{pmatrix}, \begin{pmatrix} -16/3 & -3 & 4 & -7/3 & -2/3 & -2 & 17/3 \\ 7 & 3 & -6 & 2 & 2 & 2 & -7 \\ -16/3 & -2 & 5 & -7/3 & -5/3 & -1 & 17/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -4/3 & 0 & 1 & 2/3 & -5/3 & 1 & 2/3 \\ -10/3 & -2 & 2 & -1/3 & -5/3 & -1 & 8/3 \\ 2 & 0 & -3 & 1 & 1 & 0 & -2 \end{pmatrix} \right\rangle . \tag{3.4}$$

This group is in fact PSL(2;8), of order 504 and with character table:

	1	56	56	56	56	63	72	72	72
Γ_1	1	1	1	1	1	1	1	1	1
Γ_2	7	-2	1	1	1	-1	0	0	0
Γ_3	7	1	$-p$	$-q$	$p+q$	-1	0	0	0
Γ_4	7	1	$p+q$	$-p$	$-q$	-1	0	0	0
Γ_5	7	1	$-q$	$p+q$	$-p$	-1	0	0	0
Γ_6	8	-1	-1	-1	-1	0	1	1	1
Γ_7	9	0	0	0	0	1	r	s	t
Γ_8	9	0	0	0	0	1	s	t	r
Γ_9	9	0	0	0	0	1	t	r	s

with $(p,q) := (\cos(4\pi/9), \cos(8\pi/9))$ and $(r,s,t) := (\cos(2\pi/7), \cos(4\pi/7), \cos(6\pi/7))$.

And thence follows the next imprimitive, IP_3, with generators

$$IP_3 := \left\langle \begin{pmatrix} 20/7 & 5 & 8/7 & 10/7 & -17/7 & 6 & -13/7 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1/7 & -1 & 8/7 & -4/7 & 4/7 & 0 & 1/7 \\ 0 & 3 & 0 & 1 & -2 & 2 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -16/7 & -4 & -12/7 & -8/7 & 15/7 & -5 & 9/7 \\ -4 & -7 & 0 & -1 & 3 & -7 & 1 \end{pmatrix}, \begin{pmatrix} -33/7 & -8 & -9/7 & -13/7 & 27/7 & -9 & 12/7 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -4/7 & -1 & -3/7 & -2/7 & 2/7 & -2 & -3/7 \\ -5/7 & 1 & -2/7 & 1/7 & -1/7 & 0 & -2/7 \\ 4/7 & 0 & 3/7 & 2/7 & 5/7 & 1 & -4/7 \\ 23/7 & 5 & 5/7 & 8/7 & -15/7 & 6 & -2/7 \\ 12/7 & 4 & -5/7 & -1/7 & -13/7 & 3 & -5/7 \end{pmatrix} \right\rangle . \tag{3.5}$$

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This group is isomorphic to PGL(2; 7) and is of order 336. Its characters are:

	1	21	28	42	42	42	48	56	56
Γ_1	1	1	1	1	1	1	1	1	1
Γ_2	1	1	-1	-1	-1	1	1	-1	1
Γ_3	6	-2	0	0	0	2	-1	0	0
Γ_4	6	2	0	$-\sqrt{2}$	$\sqrt{2}$	0	-1	0	0
Γ_5	6	2	0	$\sqrt{2}$	$-\sqrt{2}$	0	-1	0	0
Γ_6	7	-1	1	-1	-1	-1	0	1	1
Γ_7	7	-1	-1	1	1	-1	0	-1	1
Γ_8	8	0	2	0	0	0	1	-1	-1
Γ_9	8	0	-2	0	0	0	1	1	-1

Our fourth member is the group IP_4, generated by

$$IP_4 := \left\langle \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \right\rangle . \qquad (3.6)$$

This group IP_4 is identified as $PU(3;3)$, of order 6048. The character table is:

	1	56	63	63	63	378	504	504	504	672	756	756	864	864
Γ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Γ_2	6	-3	-2	-2	-2	2	1	1	1	0	0	0	-1	-1
Γ_3	7	-2	-1	3	3	-1	0	0	2	1	-1	-1	0	0
Γ_4	7	-2	3	$-1-2i$	$-1+2i$	1	$-1-i$	$-1+i$	0	1	$-i$	i	0	0
Γ_5	7	-2	3	$-1+2i$	$-1-2i$	1	$-1+i$	$-1-i$	0	1	i	$-i$	0	0
Γ_6	14	5	-2	2	2	2	-1	-1	1	-1	0	0	0	0
Γ_7	21	3	5	1	1	1	1	1	-1	0	-1	-1	0	0
Γ_8	21	3	1	$-3+2i$	$-3-2i$	-1	i	$-i$	1	0	$-i$	i	0	0
Γ_9	21	3	1	$-3-2i$	$-3+2i$	-1	$-i$	i	1	0	i	$-i$	0	0
Γ_{10}	27	0	3	3	3	-1	0	0	0	0	1	1	-1	-1
Γ_{11}	28	1	-4	$-4i$	$4i$	0	i	$-i$	-1	1	0	0	0	0
Γ_{12}	28	1	-4	$4i$	$-4i$	0	$-i$	i	-1	1	0	0	0	0
Γ_{13}	32	-4	0	0	0	0	0	0	0	-1	0	0	$\frac{1-i\sqrt{7}}{2}$	$\frac{1+i\sqrt{7}}{2}$
Γ_{14}	32	-4	0	0	0	0	0	0	0	-1	0	0	$\frac{1+i\sqrt{7}}{2}$	$\frac{1-i\sqrt{7}}{2}$

Finally, the largest member in our classification, is the group IP_5,

$$IP_5 := \left\langle \begin{pmatrix} -5/2 & -7 & -15/4 & -1/4 & 15/2 & -2 & 7/2 \\ -1/4 & -3 & -15/8 & -5/8 & 13/4 & -1/2 & 3/4 \\ -5/4 & -5 & -15/8 & 3/8 & 17/4 & -3/2 & 11/4 \\ -3/4 & 3 & 15/8 & -3/8 & -9/4 & -1/2 & -3/4 \\ -1 & -5 & -5/2 & -1/2 & 5 & -1 & 2 \\ 2 & 2 & 3/2 & 1/2 & -2 & 2 & -3 \\ -1/4 & -4 & -15/8 & 3/8 & 17/4 & -1/2 & 3/4 \end{pmatrix}, \begin{pmatrix} -9/4 & -4 & -15/8 & 3/8 & 21/4 & -3/2 & 7/4 \\ -5/2 & -4 & -9/4 & 1/4 & 9/2 & -2 & 5/2 \\ 3/2 & 3 & 7/4 & 1/4 & -5/2 & 1 & -5/2 \\ -1 & -3 & -1 & 0 & 2 & -2 & 3 \\ -7/4 & -3 & -13/8 & 1/8 & 15/4 & -3/2 & 5/4 \\ 1/2 & 6 & 13/4 & -1/4 & -11/2 & 1 & -5/2 \\ -5/4 & 2 & 9/8 & 3/8 & -3/4 & -1/2 & -1/4 \end{pmatrix} \right\rangle . \qquad (3.7)$$

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This IP_5 is in fact none other than $G_2(2)$, i.e., G_2 defined over the Galois field $\mathbb{F}_2$. The order is the rather formidable 12096 and the character table is

	1	56	63	126	252	252	378	504	672	1008	1008	1008	1512	1512	1728	2016
Γ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Γ_2	1	1	1	1	-1	-1	1	1	1	-1	-1	1	-1	1	1	-1
Γ_3	6	-3	-2	-2	0	0	2	1	0	$-i\sqrt{3}$	$i\sqrt{3}$	1	0	0	-1	0
Γ_4	6	-3	-2	-2	0	0	2	1	0	$i\sqrt{3}$	$-i\sqrt{3}$	1	0	0	-1	0
Γ_5	7	-2	-1	3	-3	1	-1	2	1	0	0	0	-1	-1	0	1
Γ_6	7	-2	-1	3	3	-1	-1	2	1	0	0	0	1	-1	0	-1
Γ_7	14	-4	6	-2	0	0	2	0	2	0	0	-2	0	0	0	0
Γ_8	14	5	-2	2	-2	2	2	1	-1	1	1	-1	0	0	0	-1
Γ_9	14	5	-2	2	2	-2	2	1	-1	-1	-1	-1	0	0	0	1
Γ_{10}	21	3	5	1	-1	3	1	-1	0	-1	-1	1	1	-1	0	0
Γ_{11}	21	3	5	1	1	-3	1	-1	0	1	1	1	-1	-1	0	0
Γ_{12}	27	0	3	3	-3	-3	-1	0	0	0	0	0	1	1	-1	0
Γ_{13}	27	0	3	3	3	3	-1	0	0	0	0	0	-1	1	-1	0
Γ_{14}	42	6	2	-6	0	0	-2	2	0	0	0	0	0	0	0	0
Γ_{15}	56	2	-8	0	0	0	0	-2	2	0	0	0	0	0	0	0
Γ_{16}	64	-8	0	0	0	0	0	0	-2	0	0	0	0	0	1	0

And so these above are the groups of our interest, the 7 exceptional discrete finite irreducible subgroups of G_2. Thus armed, let us now venture into the physics.

4. McKay quivers, G_2 and gauge theories

The construction of world-volume gauge theories for D-brane probes transverse to orbifold singularities was pioneered in [27] wherein the plethora of mathematical machinery in the resolution of singularities using the hyper-Kähler quotient was brought into string theory. The case then studied, as well as by subsequent works, have been on local models of non-compact Calabi-Yau manifolds. In other words the orbifolds originated from the discrete finite subgroups of the holonomy SU(2) (cf. [27, 29]), SU(3) (cf. [21, 26, 23]) and SU(4) (cf. [24, 25]).

The general methodology for retrieving the world-volume gauge data from orbifolds was outlined in [28] where the D-terms and F-terms of the gauge theory became purely dependent upon the group-theoretical properties of the orbifold, notably the group representation ring and in particular the Clebsch-Gordan composition therein. Recently, [12] and [15] discussed some abelian examples of G_2 orbifolds in this context.

We here study M-theory in the back-ground of a singular spacetime: to result in $\mathcal{N} = 1$ supersymmetry in four dimensions it is well-known that one needs to "compactify" on real sevenfolds of G_2 holonomy. Whence our singularity X will be non-compact local G_2 of the form $\mathbb{R}^7/(\Gamma \subset G_2)$ and we are studying some brane probe on X. Of course the resolution of singularities of this type, especially crepant ones to G_2 manifolds still remain an open question in mathematics [9] due largely to the want of more powerful techniques in *real* algebraic geometry. Yet let us trudge on.

4.1 D-Brane probes in type II

Following the prescription of [12]–[14], we will study the D-brane probe theories of type II originating from the parent M-theory. Indeed, we have much more knowledge of D-brane worldvolume technology than M-branes. Therefore our natural setting will be the reduction of the M-brane probe theory on the singular space X of G_2 holonomy, to a D-brane probe theory of type IIA.

As pointed out in [13], we can do so in two ways. We can reduce on an S^1 transverse to both the M-probe and to X. In this case we have X being the Higgs branch of the moduli space of the D2-brane world-volume gauge theory. Alternatively, we can reduce on an S^1 contained within X, leading to type IIA backgrounds with D6-branes and/or RR flux which may give rise to extra subtleties. We will concentrate on X as the Coulomb branch of the D2-brane theory.

Therefore the preparatory work in the above sections will be in service to the $\mathcal{N} = 1$ SUSY theory in three dimensions on the D2-probe. Mirror to this picture is the type IIB perspective of [12] wherein one has a D1-probe and the world-volume theory is $(1, 1)$ sigma model in two dimensions.

The extraction of the matter content and interactions follow the canonical methods mentioned above [27, 28, 21] and we shall see the natural emergence of the McKay quiver [33].

4.2 World-volume theories and McKay quivers

The matter content descents from the parent theory of the D-brane in flat space and the resulting bi-fundamentals are summarised by a quiver diagram whose adjacency matrix a_{ij} is determined as

$$R^{(7)} \otimes R^{(i)} = \bigoplus_j a_{ij} R^{(j)} \,, \tag{4.1}$$

where $R^{(i)}$ is the i-th irreducible representation of the orbifold group Γ and $R^{(7)}$ is the defining 7-dimensional representation which for us is a real 7×7 matrix. Indeed we choose the 7-dimensional irrep so as to guarantee that our orbifold action resides in the full G_2 and not any subgroup thereof, such as SU(3) (which would make our space essentially a Calabi-Yau threefold). We refer the reader to [28] for the details, [21] for a summary and [12] for the present guise of the derivation of (4.1).

To (4.1) shall the character tables of the previous section lend an immediate hand: we can instantly invert the equation to arrive at the quiver [21] as

$$a_{ij} = \frac{1}{g} \sum_{\gamma=1}^{r} r_\gamma \chi_\gamma \chi_\gamma^{(i)} \chi_\gamma^{(j)*} \,, \tag{4.2}$$

where $\chi_\gamma^{(i)}$ is the i-th irreducible character for the conjugacy class represented by $\gamma \in \Gamma$ and χ_γ is the character of our chosen defining 7-dimensional real irrep. Furthermore, $g = |\Gamma|$ is the order of the orbifold group Γ and r_γ is the order of the conjugacy class of γ. The sum extends over the r conjugacy classes, which by the orthogonality theorem of characters is equal to the number of irreps.

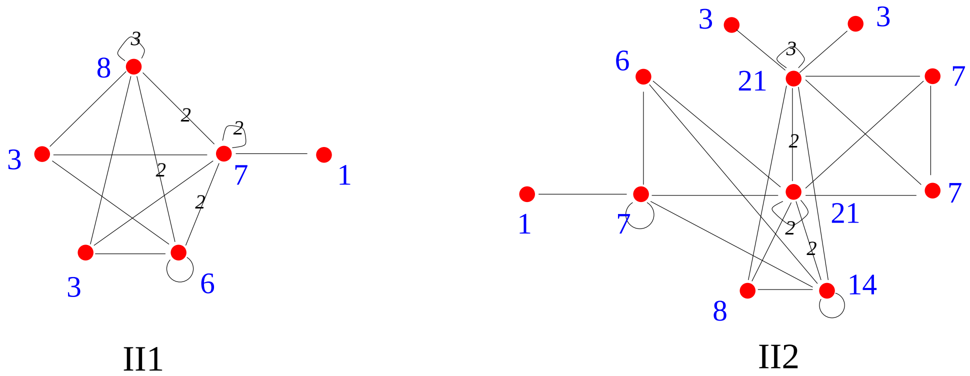

Figure 1: The quiver diagram with respect to the fundamental 7 for the 2 irreducible imprimitive discrete subgroups of G_2.

Therefore standard results dictate that if we have n parallel coincident branes (in the regular representation $n = Ng$), then from the parent $\mathrm{U}(n)$ SYM would result a daughter gauge theory which has gauge group $\prod_i^r \mathrm{U}(Nn_i)$ with a_{ij} bifundamentals transforming in the $\mathrm{U}(Nn_i) \times \mathrm{U}(Nn_j)$ factor.

We see of course, that $a_{ji} = (1/g)\sum r_\gamma \chi_\gamma \chi_\gamma^{(j)} \chi_\gamma^{(i)*}$ which since the adjacency matrix has integer entries must equal to $a_{ji}^* = (1/g)\sum r_\gamma \chi_\gamma^* \chi_\gamma^{(j)*} \chi_\gamma^{(i)}$. The latter is equal to a_{ij} precisely because our defining representation is real and thus $\chi_\gamma^* = \chi_\gamma$. Hence $a_{ji} = a_{ij}$ and our quivers are symmetric. In other words the reality of our singularity in the sense that the orbifold is a real algebraic variety compels us not to have chiral matter and we have a "non-chiral" $\mathcal{N} = 1$ theory in three dimensions.[2] In order to arrive at chiral fields, one must use the complex 7-dimensional representation for $R^{(7)}$, yet this is geometrically less clear and complications shall arise as to how one finds a real G_2 locus in the complex 7-dimensional quotient.

The McKay quivers of (4.2) nevertheless provides us with an interesting class of non-oriented finite graphs. We present them in figure 1 for the imprimitives and figure 2 for the primitives. The labels of the nodes (in blue) are the n_i's in the abovementioned product gauge group $\prod_i^r \mathrm{U}(Nn_i)$. In the case of $\Gamma \subset \mathrm{SU}(2)$ these labels, by virtue of the McKay Correspondence [33], are precisely the dual coxeter numbers of of the affine ADE Dynkin diagrams. Moreover, each edge in the graph is a bi-directional arrow due to the symmetry (non-chirality) of the adjacency matrix; multiplicities of these arrows are indicated thereupon.

As a parting digression let us briefly comment on some implications of these graphs. Indeed, as brane-probe theories, each graph corresponds to an $\mathcal{N} = 1$ gauge theory whose superpotential could also be computed using the Clebsch-Gordan coefficients of the respective groups in the manner of [28].

[2]We are of course being cavalier with the word chiral which for phenomenological purposes are of interest to four dimensional theories; by non-chiral here we merely mean non-oriented quiver and ergo symmetric a_{ij}.

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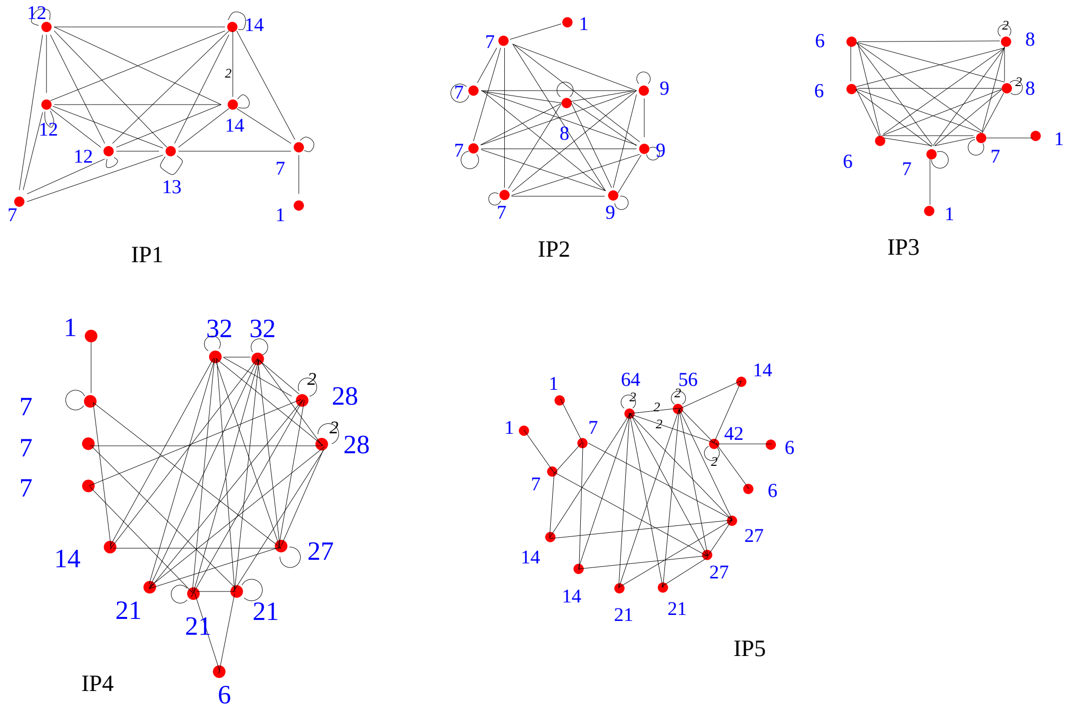

Figure 2: The quiver diagram with respect to the fundamental $\mathbf{7}$ for the 5 irreducible primitive discrete subgroups of G_2.

Furthermore, it was conjectured in [21] and addressed further in [31, 32, 34] (see [11] for some review) that string orbifolds provides some type of generalised McKay Correspondence between the representation ring of discrete subgroups of $\mathrm{SU}(n)$ and the fusion ring of $\widehat{su(n)}$ Wess-Zumino-Witten models at least for $n = 2, 3, 4$ where D-brane probe technology is applicable. So these were the cases for Calabi-Yau orbifolds, now we have G_2-orbifolds and M-theory; could there be similar relations to $\widehat{g}_2$ WZW models? Indeed, some exceptional cases for the latter model were found at levels 3 and 4 [35] while the complete classification is still in want; could these perhaps be in correspondence with the quivers thus far presented?

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Ad Catharinae Sanctae Alexandriae et Ad Majorem Dei Gloriam ...

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